Southwest State University
Department of computer sciences

ENUMERATING THE TRANSVERSALS FOR DIAGONAL LATIN SQUARES OF SMALL ORDER

Eduard I. Vatutin
Stepan E. Kochemazov
Oleg S. Zaikin
Sergey Yu. Valyaev

Petrozavodsk, 2017

What is Latin squares?

$$A = \left\| a_{ij} \right\|$$

$$i,\ j = \overline{1,\ N}$$

$$N = \left| S \right|$$

$$S = \{0, 1, 2, ..., N-1\}$$

$$\forall i,\ j,\ k = \overline{1,\ N},\ j \neq k : \left(a_{ij} \neq a_{ik}\right) \wedge \left(a_{ji} \neq a_{ki}\right)$$

$$\forall i,\ j = \overline{1,\ N}, i \neq j : \left(a_{ii} \neq a_{jj}\right) \wedge \left(a_{N-i+1,\ N-i+1} \neq a_{N-j+1,\ N-j+1}\right)$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 9 & 4 & 3 & 6 & 7 & 5 & 0 & 8 \\
2 & 9 & 3 & 1 & 7 & 0 & 5 & 8 & 4 & 6 \\
3 & 4 & 1 & 2 & 8 & 7 & 9 & 6 & 5 & 0 \\
4 & 3 & 5 & 9 & 2 & 1 & 8 & 0 & 6 & 7 \\
5 & 6 & 4 & 8 & 1 & 2 & 0 & 9 & 7 & 3 \\
6 & 5 & 8 & 7 & 0 & 3 & 2 & 1 & 9 & 4 \\
7 & 8 & 6 & 0 & 9 & 4 & 1 & 2 & 3 & 5 \\
8 & 7 & 0 & 5 & 6 & 9 & 3 & 4 & 1 & 2 \\
9 & 0 & 7 & 6 & 5 & 8 & 4 & 3 & 2 & 1
\end{pmatrix}$$

Normalized LS of order 10

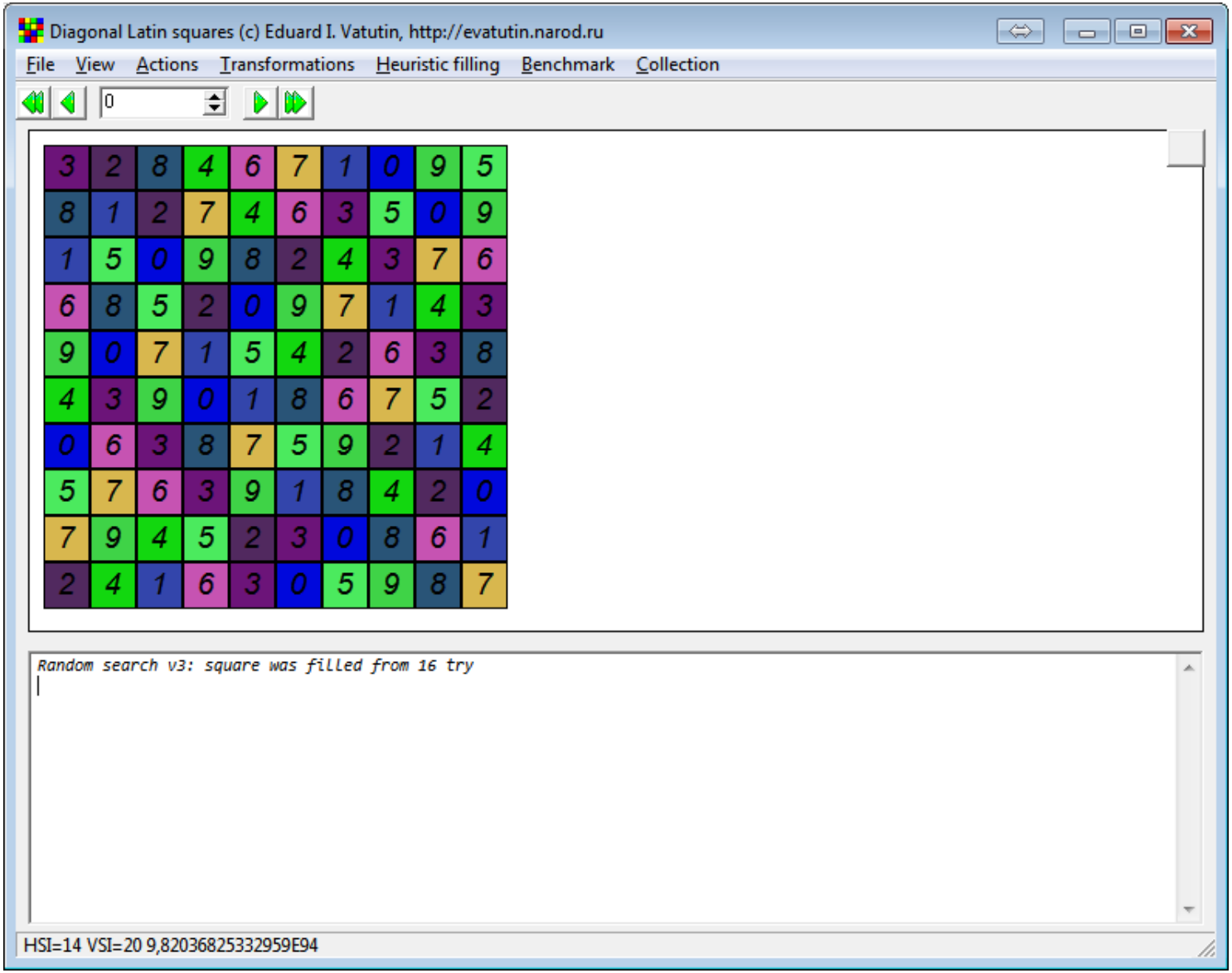
$$N! \times (N-1)!$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
7 & 2 & 4 & 9 & 0 & 6 & 5 & 1 & 3 & 8 \\
8 & 3 & 6 & 7 & 5 & 9 & 0 & 2 & 4 & 1 \\
2 & 6 & 8 & 5 & 1 & 7 & 4 & 0 & 9 & 3 \\
5 & 8 & 9 & 1 & 7 & 0 & 3 & 4 & 6 & 2 \\
9 & 4 & 1 & 2 & 8 & 3 & 7 & 6 & 0 & 5 \\
4 & 7 & 5 & 6 & 9 & 1 & 8 & 3 & 2 & 0 \\
3 & 0 & 7 & 8 & 2 & 4 & 1 & 9 & 5 & 6 \\
6 & 5 & 0 & 4 & 3 & 2 & 9 & 8 & 1 & 7 \\
1 & 9 & 3 & 0 & 6 & 8 & 2 & 5 & 7 & 4
\end{pmatrix}$$

Normalized DLS of order 10

$$(N-1)!$$

Lets try to get diagonal Latin square!



- http://evatutin.narod.ru/evatutin_LsEdit.7z

It is simple! With the right approach...

Approaches:

- Brute Force, backtracking, SAT — **0,01 DLS/s** (2014)
- diagonal filling — **28 DLS/s**
- out of order filling cells with |S|=1 — **15 000 DLS/s**
- fast check for sets of available values — **38 000 DLS/s**
- early clipping for cells with |S|=0 — **101 000 DLS/s**
- variable order of filling the cells — **240 000 DLS/s**
- special program implementation with N^2 nested loops — **340 000 DLS/s**
- use the principal of minimum abilities — **790 000 DLS/s**
- clipping for selected depth only — **1 100 000 DLS/s**
- use the formulas for magic squares — **1 800 000 DLS/s**
- use the bits arithmetic magic — **6 600 000 DLS/s** (2016)

Pace increased by **8 orders** using algorithmic optimization without parallelization!

Results: number of DLS of order N<10

A274171 (Number of diagonal Latin squares of order n with first row 1..n)
1, 0, 0, 2, 8, 128, 171200, 7447587840, 505699465350758 4

A274806 (Number of diagonal Latin squares of order n)
1, 0, 0, 48, 960, 92160, 862848000, 300286741708800, 1835082219864832081920

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

$$L_{10} \simeq (7{,}6 \div 10{,}9) \cdot 10^{22}$$

~250 000 years at Gerasim@Home distributed computing project
~1 year at 1 PFLOP/s supercomputer (who can help us? :))

- Gerasim@Home (~500 PCs, ~3 months, 2–5 TFLOP/s), http://gerasim.boinc.ru
- Matrosov academician computing cluster (~500 24/7 CPU cores, ~3 months)
- https://oeis.org (Online Encyclopedia of Integer Sequences, OEIS)

Searching for pairs of ODLS of order 10

L. Euler expected that for N=10 ODLS doesn't exist
First pair — Parker et al., 1960

0	1	2	3	4	5	6	7	8	9
1	2	0	4	3	7	9	8	5	6
7	3	5	9	0	4	8	6	2	1
3	5	6	8	9	0	4	1	7	2
4	9	7	2	6	8	1	5	0	3
5	8	4	6	7	1	3	2	9	0
8	4	9	1	2	3	7	0	6	5
6	7	3	0	1	2	5	9	4	8
9	0	1	5	8	6	2	4	3	7
2	6	8	7	5	9	0	3	1	4

0	1	2	3	4	5	6	7	8	9
7	5	1	9	2	8	0	4	6	3
1	0	3	4	6	7	5	2	9	8
9	8	4	7	5	2	1	0	3	6
6	7	9	0	8	3	2	1	5	4
4	6	5	1	0	9	8	3	2	7
2	3	8	5	1	6	4	9	7	0
5	2	7	8	3	4	9	6	0	1
3	4	6	2	9	0	7	8	1	5
8	9	0	6	7	1	3	5	4	2

SAT@Home, 04.2015

0	1	2	3	4	5	6	7	8	9
4	9	0	8	5	6	3	1	2	7
2	5	7	9	6	4	0	8	1	3
9	0	4	6	8	7	1	5	3	2
6	7	5	2	1	3	8	0	9	4
1	8	3	5	7	2	9	6	4	0
7	3	1	0	9	8	4	2	6	5
8	2	6	4	0	9	5	3	7	1
3	4	8	1	2	0	7	9	5	6
5	6	9	7	3	1	2	4	0	8

0	1	2	3	4	5	6	7	8	9
6	5	9	7	0	8	2	3	1	4
4	7	1	2	3	9	8	0	6	5
1	2	0	4	5	3	7	6	9	8
2	6	8	0	9	4	1	5	3	7
8	4	6	9	2	7	0	1	5	3
5	0	4	6	8	2	3	9	7	1
9	3	5	1	7	6	4	8	0	2
7	8	3	5	6	1	9	4	2	0
3	9	7	8	1	0	5	2	4	6

Gerasim@Home, 04.2017

Present for citerra, 2017 :)

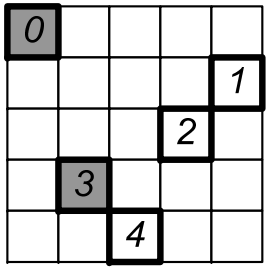
Searching for ODLS: approaches

- Brute Force + backtracking + clippings + ordering + ... (very long)
- SAT (some tens of hours, long)
- filling by pairs of elements [a_{ij}, b_{ij}] (long)
- using transversals (fast) – **<u>200 – 800 DLS/s</u>** for different algorithms!

a)

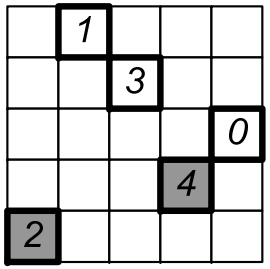
0	1	2	3	4
4	2	3	0	1
3	4	1	2	0
1	3	0	4	2
2	0	4	1	3

b)



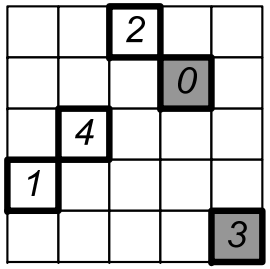
$T^{(d)}_1=\{a_{11}, a_{25}, a_{34}, a_{42}, a_{53}\}$

c)



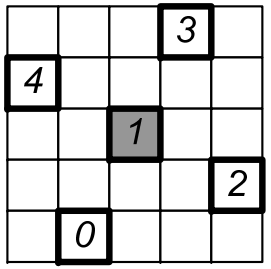
$T^{(d)}_2=\{a_{12}, a_{23}, a_{35}, a_{43}, a_{51}\}$

d)



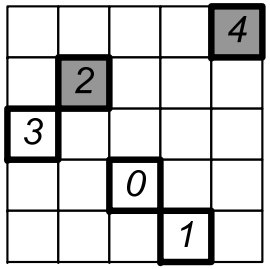
$T^{(d)}_3=\{a_{13}, a_{24}, a_{32}, a_{41}, a_{55}\}$

e)



$T^{(d)}_4=\{a_{14}, a_{21}, a_{33}, a_{45}, a_{52}\}$

f)



$T^{(d)}_5=\{a_{15}, a_{22}, a_{31}, a_{43}, a_{54}\}$

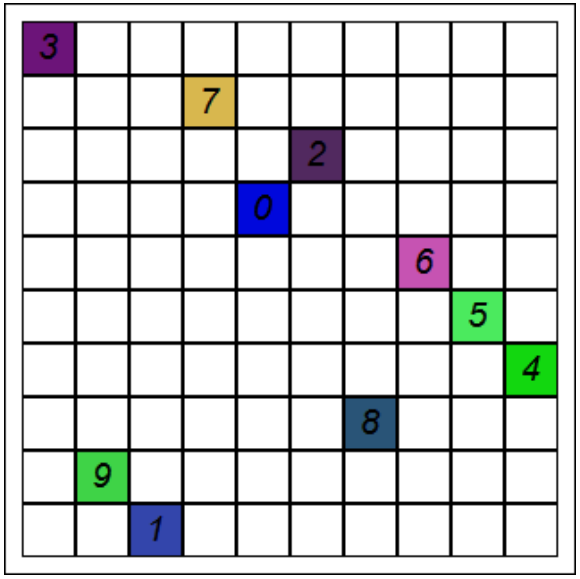
Crossing and orthogonal transversals

3	2	8	4	6	7	1	0	9	5
8	1	2	7	4	6	3	5	0	9
1	5	0	9	8	2	4	3	7	6
6	8	5	2	0	9	7	1	4	3
9	0	7	1	5	4	2	6	3	8
4	3	9	0	1	8	6	7	5	2
0	6	3	8	7	5	9	2	1	4
5	7	6	3	9	1	8	4	2	0
7	9	4	5	2	3	0	8	6	1
2	4	1	6	3	0	5	9	8	7

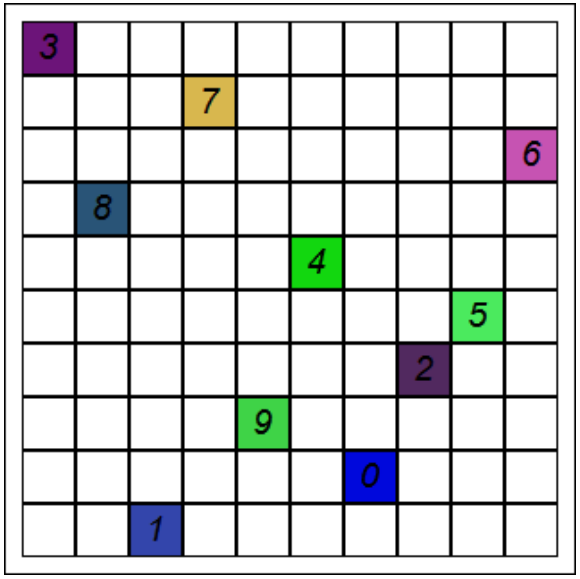
$$T_1 \cap T_2 = \{3, 7, 5, 1\}$$

$$T_1 \perp T_3 \left(T_1 \cap T_2 = \varnothing\right)$$

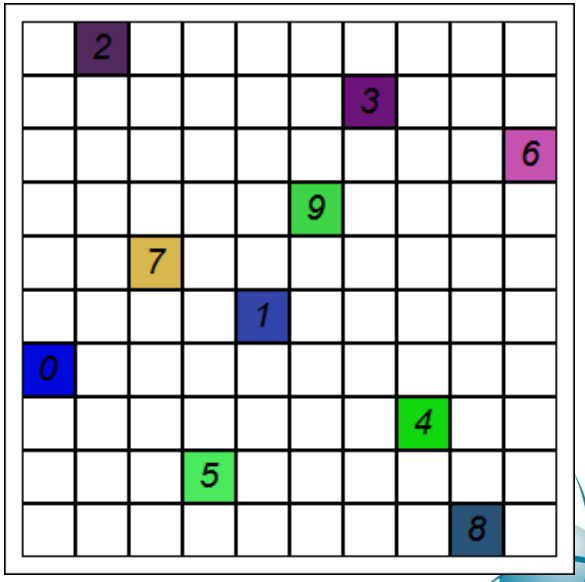
$$T_2 \cap T_3 = \{6\}$$



Transversal 1



Transversal 2



Transversal 3

Some combinatorial characteristics of DLS

Minimal and maximal number of transversals:

1, 0, 0, 8, 3, 32, 7, 8 (N < 9), evatutin, veinamond, 2017

1, 0, 0, 8, 15, 32, 133, 384 (N < 9), evatutin, veinamond, 2017

Minimal and maximal number of *diagonal* transversals:

1, 0, 0, 4, 1, 2, 0, 0 (N < 9), evatutin, veinamond, 2017

1, 0, 0, 4, 5, 6, 27, 120 (N < 9), evatutin, veinamond, 2017

Bolded red values calculated using Gerasim@Home project

(1 week with 1,5 TFLOP/s real performance)

Sequences was reviewed and added to OEIS by our collective!

Related combinatorial characteristics of DLS

DLS main classes amount:

1, 0, 0, 1, 2, 2, 972, 4 873 096 (N < 9), evatutin, whitefox, 2017
http://forum.boinc.ru/default.aspx?g=posts&m=87549#post87549

Number of the normalized symmetric and double symmetric DLS:

0, 2, 64, 3 612 672 (N < 8), evatutin, 2017
0, 2, 0, 15 780 (N < 8), evatutin, 2017

Number of reduced pairs of orthogonal diagonal Latin squares:

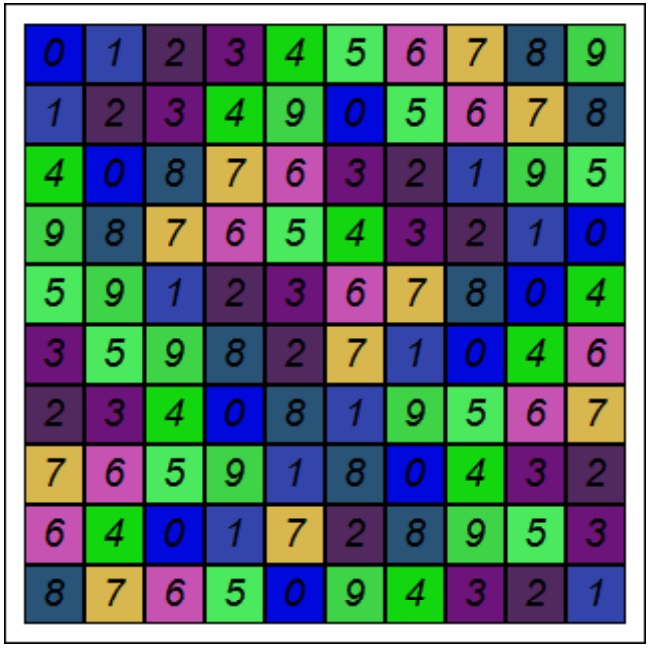
1, 0, 0, 2, 4, 0, 320 (N < 8), evatutin, 2017

Maximum number of orthogonal diagonal Latin squares for one diagonal Latin square:

1, 0, 0, 1, 1, 0, 3, 824 (N < 9), hoarfrost, evatutin, 2017

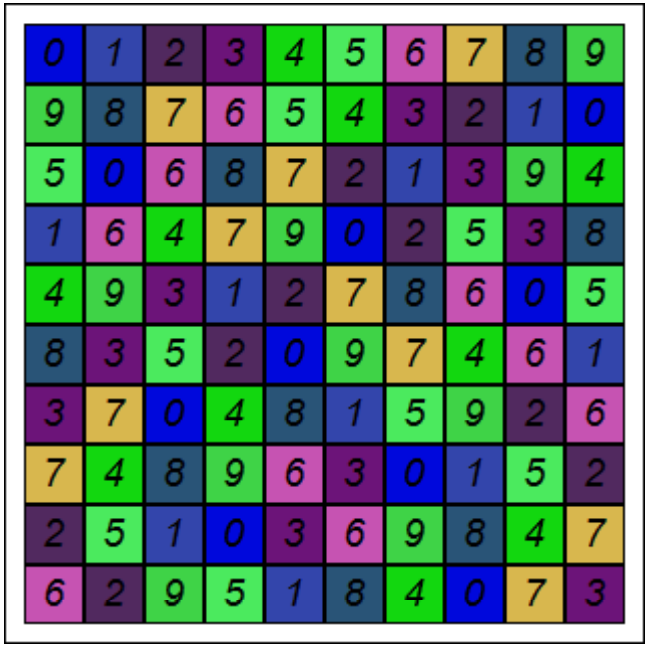
Sequences was also reviewed and added to OEIS by our collective!

Searching for triples of MODLS of order 10: they are exists?



0	1	2	3	4	5	6	7	8	9
1	2	3	4	9	0	5	6	7	8
4	0	8	7	6	3	2	1	9	5
9	8	7	6	5	4	3	2	1	0
5	9	1	2	3	6	7	8	0	4
3	5	9	8	2	7	1	0	4	6
2	3	4	0	8	1	9	5	6	7
7	6	5	9	1	8	0	4	3	2
6	4	0	1	7	2	8	9	5	3
8	7	6	5	0	9	4	3	2	1

Orthogonality characteristic 74,
citerra
(**world record, 2016**)



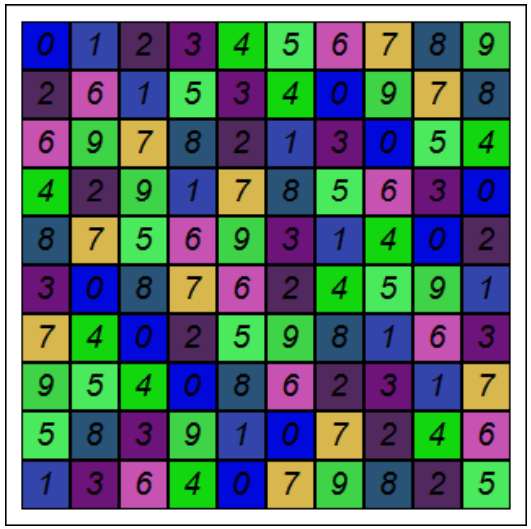
0	1	2	3	4	5	6	7	8	9
9	8	7	6	5	4	3	2	1	0
5	0	6	8	7	2	1	3	9	4
1	6	4	7	9	0	2	5	3	8
4	9	3	1	2	7	8	6	0	5
8	3	5	2	0	9	7	4	6	1
3	7	0	4	8	1	5	9	2	6
7	4	8	9	6	3	0	1	5	2
2	5	1	0	3	6	9	8	4	7
6	2	9	5	1	8	4	0	7	3

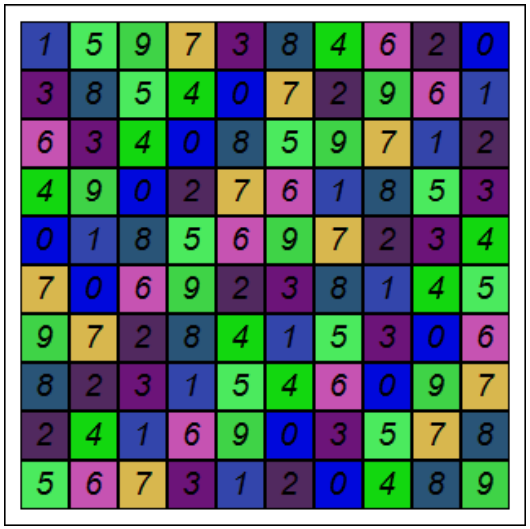
Orthogonality characteristic 74,
evatutin (2017)

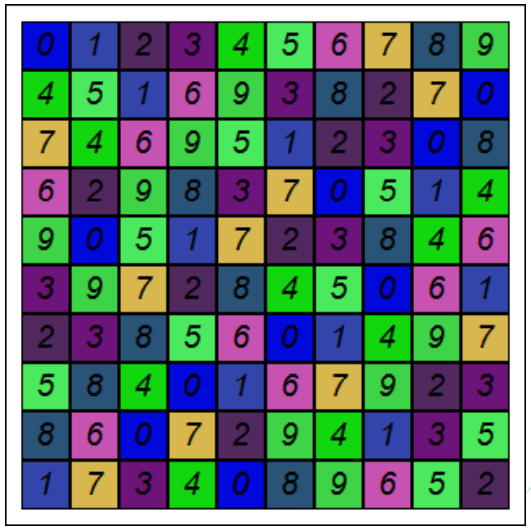
- Characteristic value can be increased? It is open question...
- Decisions are differ?

Decisions are same: concept of the canonical forms (CFs)

- Equivalence relation split objects by isomorphism classes
- DLS has different separations on the isomorphism classes (paratopes, isomorphs, etc.)
- Main classes — equivalence by amount and structure of transversals set
- Transformations:
  - normalize
  - rotations
  - symmetric flips
  - transpose
  - permutations of the symmetric rows and columns (M-transformations by Yu. Chebrakov for magic squares)







Search and collecting of the unique ODLS CFs

On 17.08.2017 collection includes **233 809** unique ODLS CFs (1 CF — isomorphism class of 7 680 or 15 360 DLS). Available for free access at:

http://forum.boinc.ru/default.aspx?g=posts&m=88700
http://forum.boinc.ru/resource.ashx?a=2940

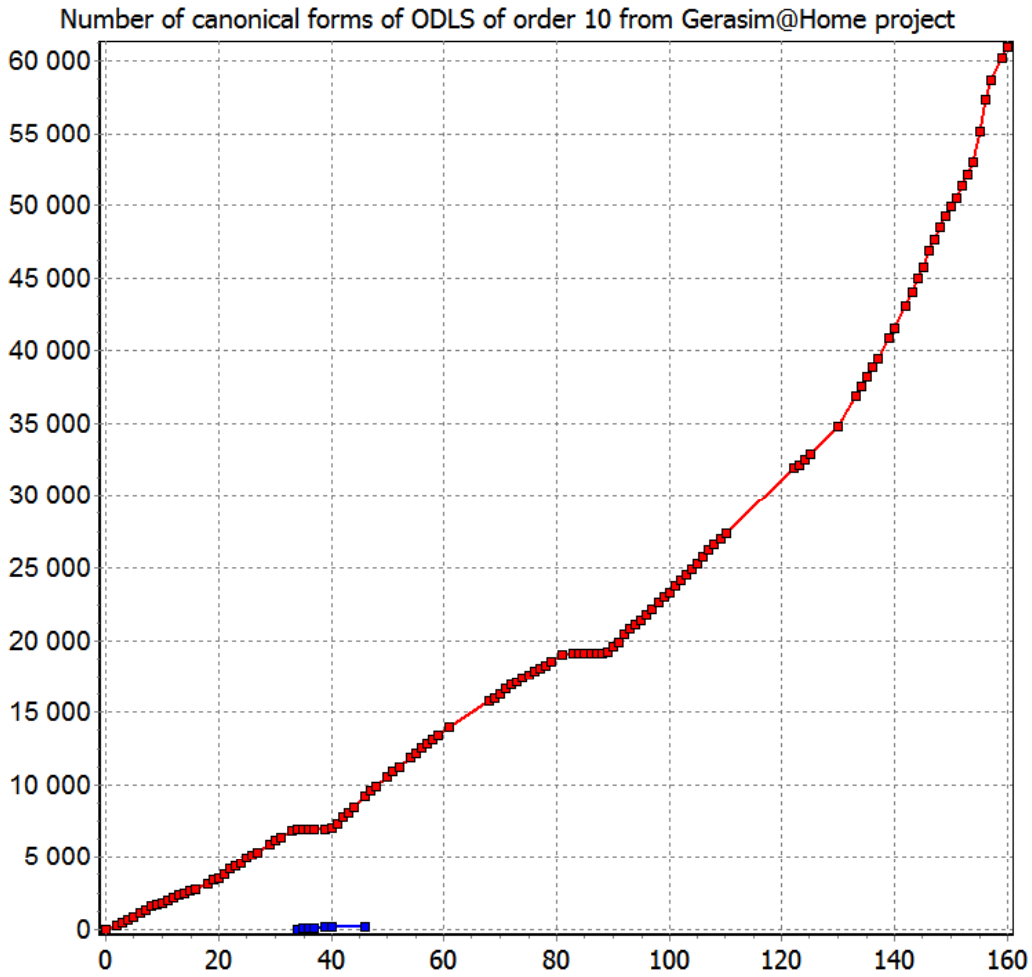
From collection items:

- **6** CFs (Parker, Brown, 1960 – 2000)
- **144** CFs (Nauchnik + SAT@Home volunteers, 2013 – 2015)
- **2 739** symmetric CFs (citerra, evatutin + Gerasim@Home volunteers, from 2016)
- **1 227** «Brown» CFs (Brown, whitefox, citerra, 2016 – 2017)
- **30 502** SODLS CFs (H. White + whitefox, 2017)
- **60 950** CFs from Gerasim@Home project (evatutin + Gerasim@Home volunteers, 2017), **~ 1 000 CFs per day**
- **137 931** CFs from ODLK@Home project (Progger + ODLK@Home volunteers, 2017)
- **~300** CFs (different sources, 2000)

Triple of MODLS or pseudo triple with orthogonality characteristic greater than 74 not found!

Getting ODLS CFs within Gerasim@Home project

Strategy of search: getting source square (random generator, symmetric random generator), try to get orthogonal square, add the unique CF to collection



- Brute Force with bits arithmetic (03.2017)
- DLX v1, array (04.2017)
- DLX v2, pointers (05.2017)
- SN DLS (SCFs) (08.2017)

Special types of squares and its properties

0	1	2	3	4	5
4	2	0	5	3	1
5	4	3	2	1	0
2	5	4	1	0	3
3	0	1	4	5	2
1	3	5	0	2	4

0	1	2	3	4	5
4	2	5	0	3	1
3	5	1	2	0	4
5	3	0	4	1	2
2	4	3	1	5	0
1	0	4	5	2	3

Symmetric DLS examples

0	1	2	3	4	5	6	7	8	9
9	8	7	6	5	4	3	2	1	0
8	0	6	7	9	3	4	5	2	1
1	2	5	4	3	9	7	6	0	8
7	9	3	1	2	6	8	0	4	5
6	5	1	9	0	7	2	8	3	4
5	4	0	8	6	2	1	3	9	7
3	6	4	5	1	8	0	9	7	2
4	3	8	2	7	0	9	1	5	6
2	7	9	0	8	1	5	4	6	3

String-inverse DLS

0	1	2	3	4	5	6	7	8	9
5	9	6	4	8	1	3	0	2	7
9	0	1	8	6	2	7	4	5	3
4	6	5	2	0	7	8	3	9	1
2	4	9	7	3	6	1	8	0	5
3	7	8	9	5	4	0	2	1	6
7	8	3	0	2	9	5	1	6	4
8	5	7	1	9	0	4	6	3	2
6	3	4	5	1	8	2	9	7	0
1	2	0	6	7	3	9	5	4	8

SODLS

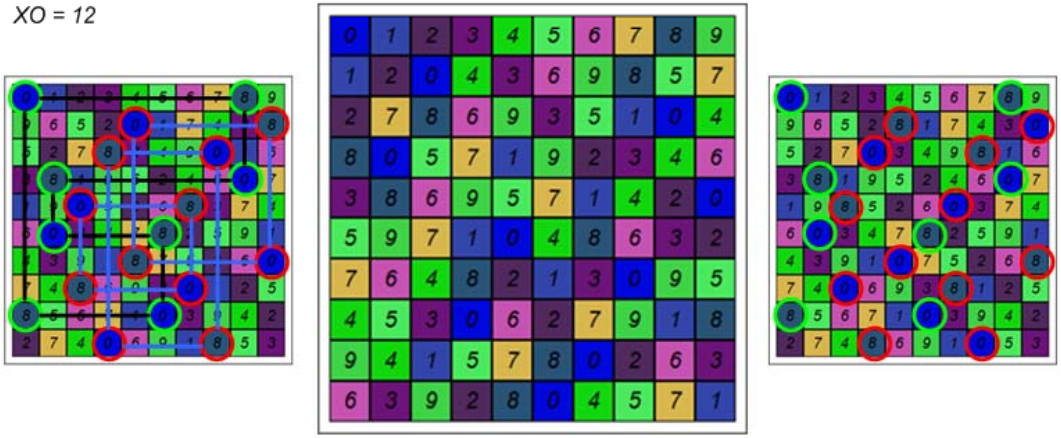
0	1	2	3	4	5	6	7
6	2	3	7	0	4	5	1
4	5	1	0	7	6	2	3
5	6	7	4	3	0	1	2
7	3	6	2	5	1	4	0
2	7	4	1	6	3	0	5
3	0	5	6	1	2	7	4
1	4	0	5	2	7	3	6

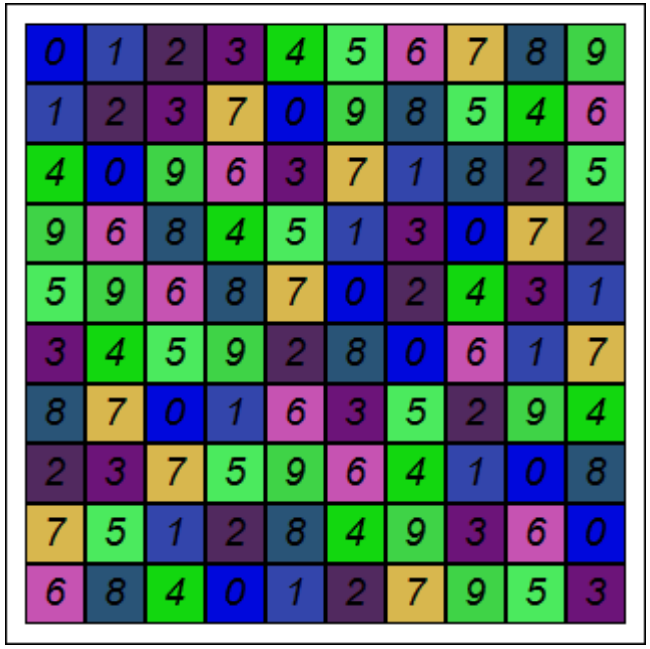
Double symmetric DLS

Classification of DLS of order 10 by number of orthogonal squares

From all ODLS CFs we have (on 02.04.2017):

- **1:0** — absolute majority (~1 by 30 000 000 DLS, bachelor)
- **1:1** — >200 000 (most of known ODLS)
- **1:2** — >3 100 (symmetric and not!)
- **1:3** — 1 (whitefox, 2017)
- **1:4** — 205
- **1:6** — 6
- **1:8** — 4
- some else?...





«Treshka» from whitefox

Different types of classification are under development (fish, rhombus, ...), classification process needs to be automated!

GPU implementation of Euler-Parker approach (I. Shutov)

10x times faster than single threaded CPU implementation

Based on:

- parallel processing of different squares on different SMXs
- parallel building of sets of transversals based on 10 cells (with WARP)
- efficient use of the CUDA shared memory (for square being processed) and register memory (for additional data structures)

Advantages and disadvantages:

- faster than single threaded CPU (~2500 CUDA cores per GPU working in parallel!)
- slower that peak abilities of GPU (for example, molecular dynamics or N-body problem — 600x times faster than CPU)

Problems:

- Recursive algorithm (but iterative implementation)
- irregular if's patterns (difficult to effective execution of WARPs)
- irregular memory accesses

Dancing Links X algorithm (DLX, D. Knuth, 2000)

At now 4x times faster than bits arithmetic approach

Based on fast decision of the **exact cover problem** solving

Generating of DLS —

$N^3 \times 3N^2+2N$

Generating of normalized DLS —

$N^3–N^2 \times 3N^2+2N$

Transversals set building —

$N^2 \times 3N+2$

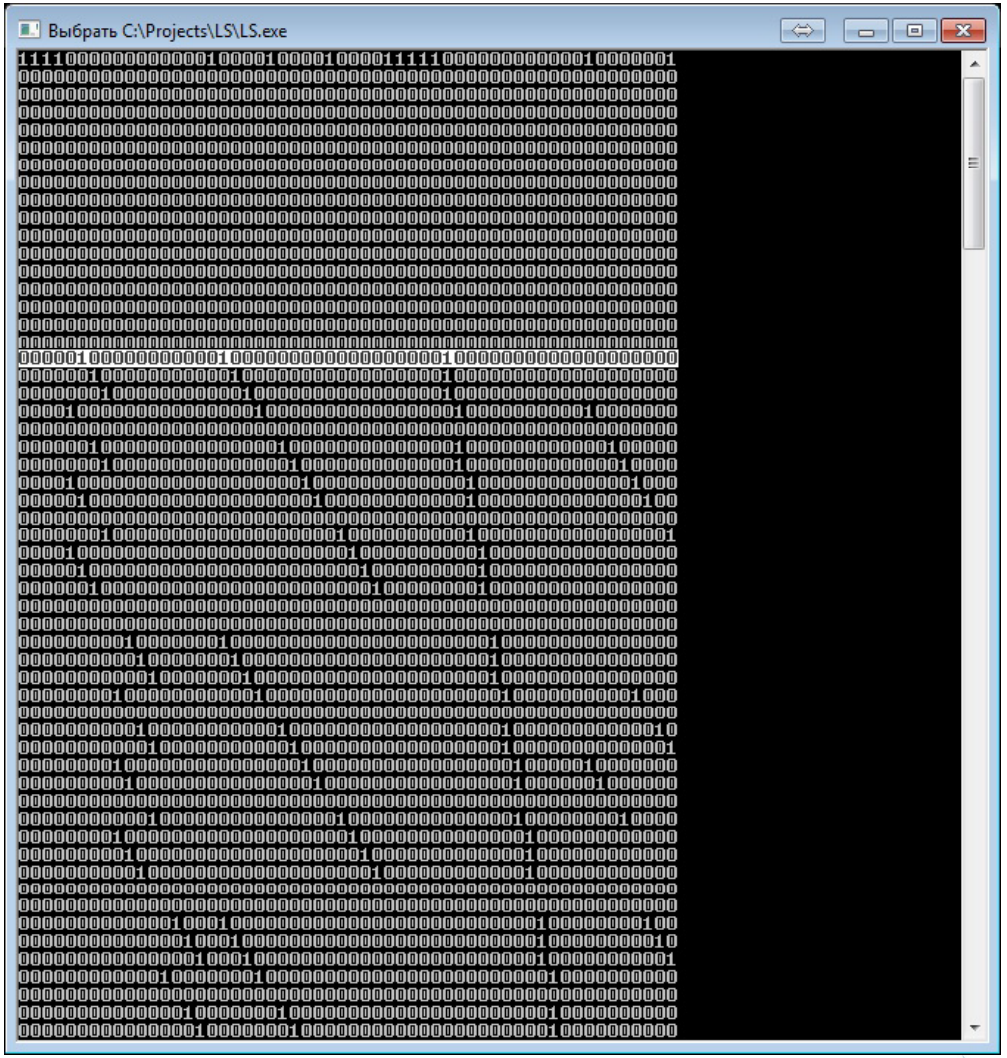
Getting of ODLS (directly) —

$N^3 \times 4N^2+2N$

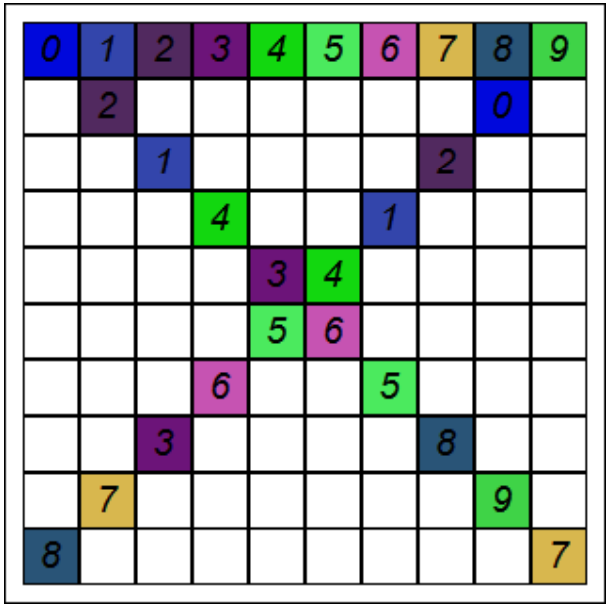
Getting of ODLS using transversals set (efficient) —

$T \times N^2$

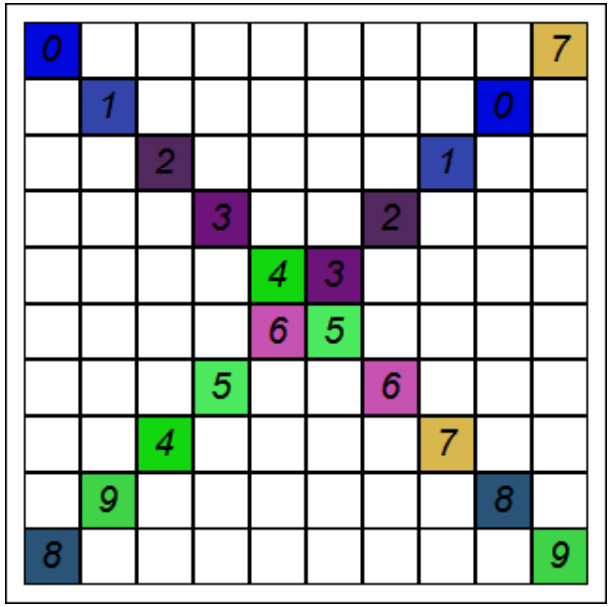
Can be implemented on GPU? Shared memory volume restriction? Irregular memory access patterns?



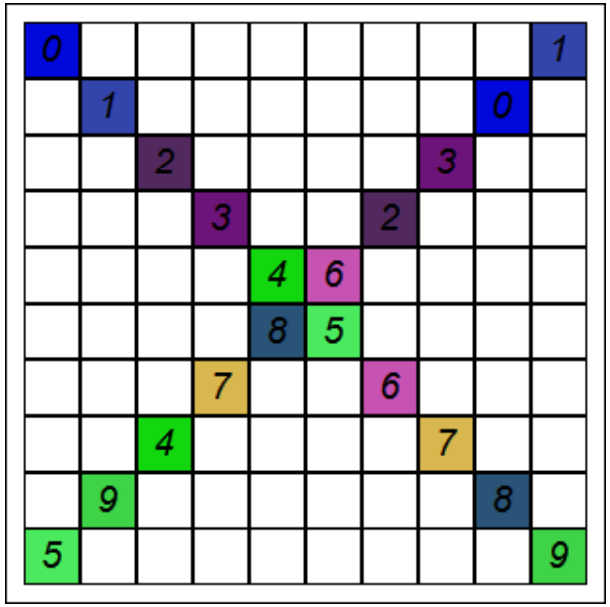
Strong normalized DLS



2 723 433 984 different fillings
(huge amount)



440 192 different fillings only
(~ 6000x times less)

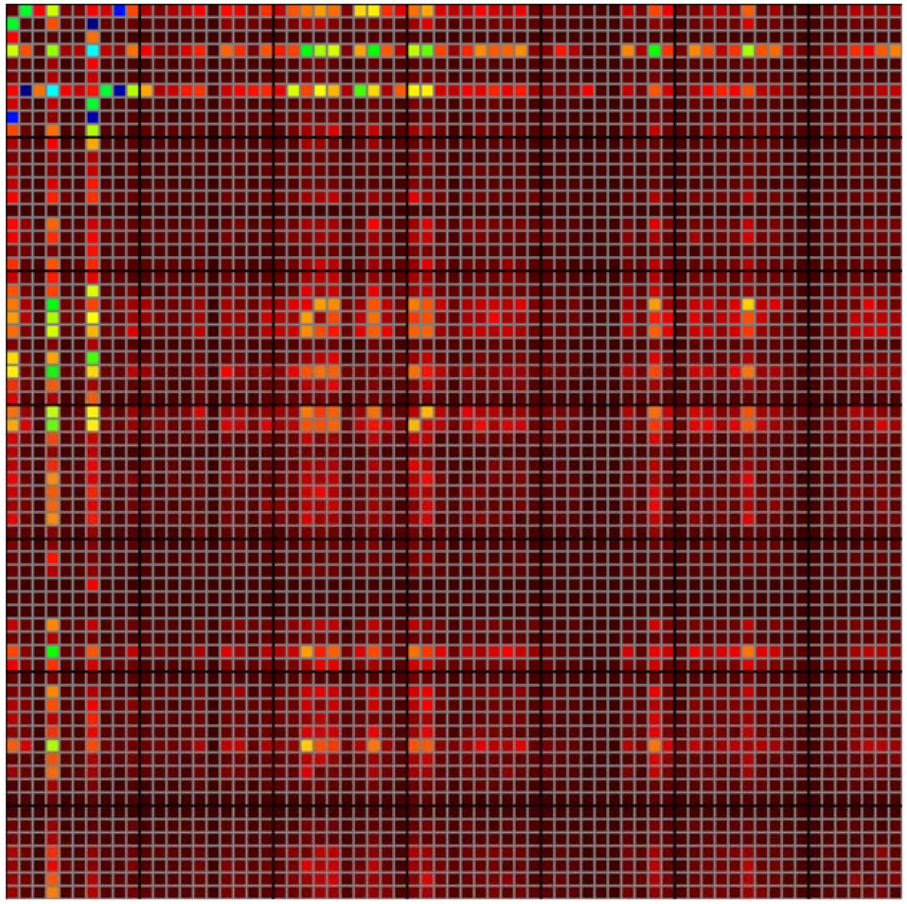


67 different fillings only!!!
(~ 6500x times less)

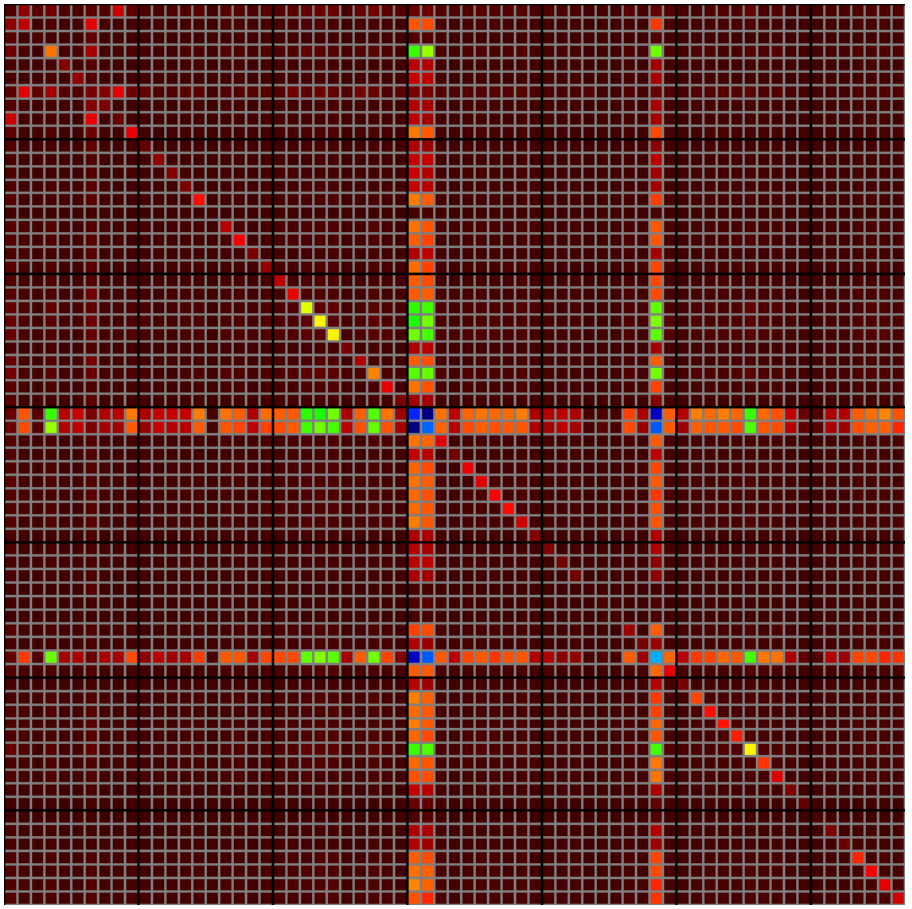
- ODLS@Home – lines 1034689527, 1034689572 and 1204678953 (CFs only, multiplicity 15 360)
- Gerasim@Home – lines 1032674598 (mult. 320x, few?), 1032675894 (mult. 15 360), random select after that?

- 67 lines of SCFs with different properties (multiplicity, CFs density, ODLS CFs density, LS-by-DLS canonization features)

Cross correlation of SCFs



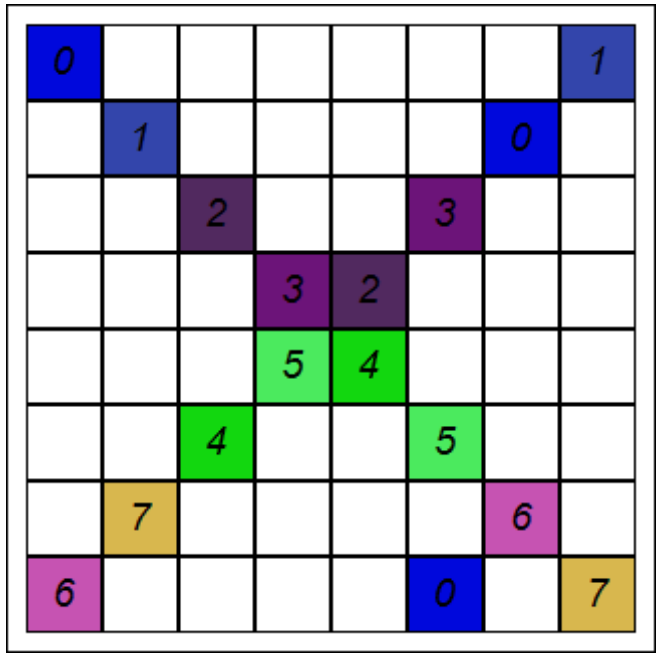
Gerasim@Home, ~ 60 000 CFs, random + symmetry + Browns



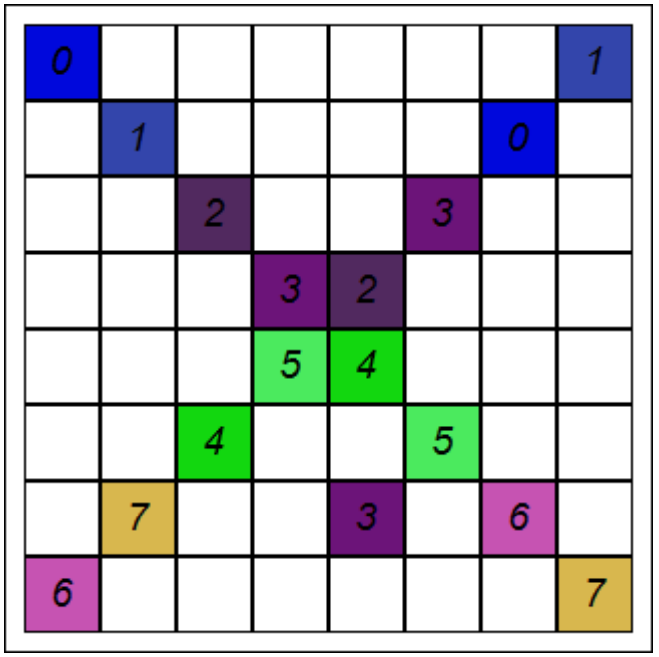
Total, ~ 230 000 CFs

- different multiplicity of lines
- SODLS

Separating of SCFs to isomorphism classes



213 920 different DLSs



213 920 different DLSs
also, why?
Same isomorphism class!

- require to change order of filling of lexicographic strings (simple, but CFs are differ);
- require to work with partially filled CFs (new algorithm, implementation, optimization);
- can be used for fast enumerating (10x – 100x times faster).

Related work

Collecting CFs and new combinatorial structures search:

- triple of MODLS (it is exists?)
- different structures?

GPU implementation of transversal and cover algorithms:

- Euler-Parker approach – need to deploy to Gerasim@Home project
- DLX – need to develop (it is faster on CPU, and what about GPU?)

Enumeration problems (OEIS):

- expanding current sequences
- enumerating DLS and ODLS of special kind (string-inverse, symmetric, ...) and its CFs
- special procedures for special types of squares (symmetric DLX, ...)

Pseudo triples:

- 3 kinds of pseudo triples, only 1 was investigated in details
- special approaches (optimization problem, ACO?, ...)

Thank you for your attention!

Thanks to all the volunteers who took part in the Gerasim@home project!

WWW: http://evatutin.narod.ru, http://gerasim.boinc.ru
E-mail: evatutin@rambler.ru
LJ: http://evatutin.livejournal.com
Skype: evatutin